BIBLIOGRAPHY

1. Landau, L. D. and Lifshits, E. M., Mechanics of Continuous Media. Moscow, Gostekhteorizdat, 1944.
2. Squire, H. B., The round laminar jet. Quart. J. Appl. Math., Vol. 4, №3, 1951.
3. Iatseev, V. I., On a class of exact solutions of the equations of motion of a viscous fluid. Zh. Eksperim. i Teor. Fiz., Vol. 20, №11, 1950.
4. Wu, Ching-Sheng, A class of exact solutions of the magnetohydrodynamic Navier-Stokes equations. Quart. J. Mech. Appl. Math., Vol. 14, №1, 1961.
5. Williams, J. C., Conical nozzle flow with velocity slip and temperature jump. AIAA J. Vol. 5, №12, 1967.
6. Byrkin, A. P., Exact solution of the Navier-Stokes equations for a compressible gas. PMM Vol. 33, №1, 1969.

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STEADY FLOW OF A VISCOELASTIC FLUID IN A CHANNEL WITH PERMEABLE WALLS

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The flow of a Maxwellian fluid in a plane channel whose boundaries move at given velocities is considered. The problem need not have a continuous solution in the event of mass flow through the channel boundaries. The factors occasioning a discontinuous solution are discussed. The three-constant Oldroyd model is used as an example to analyze the possible discontinuity structure of the initial two-constant model.

1. Let us assume that the flow of a viscoelastic fluid depends on the single coordinate z and that the behavior of the fluid is described by the rheological equations of Oldroyd's "contravariant" model [1]

$$p_{ij} = -p\delta_{ij} + T_{ij}, \qquad T_{ij} + \lambda_1 T_{ij}^{\cdot} = 2\eta e_{ij}$$
$$T_{ij}^{\cdot} = \partial T_{ij} / \partial t + v_k T_{ij,k} - v_{i,k} T_{kj} - v_{j,k} T_{ik} \tag{1.1}$$

Let the velocity vector be of the form $\mathbf{V} = (v_x,\ 0,\ v_0)$, let the tensor T_{ij} have the nonzero components T_{xx}, T_{xz}, T_{zz}, and let the longitudinal pressure gradient and external body forces equal zero. In the steady-flow case which we shall consider the continuity equation implies that $v_0 = \text{const}$; the equations of motion and relations (1.1) yield the following system of equations closed with respect to $v_x, T_{xz}, T_{xx}, T_{zz}$:

$$\rho v_0 \frac{dv_x}{dz} = \frac{dT_{xz}}{dz}, \qquad T_{xz} + \lambda_1 \left(v_0 \frac{dT_{xz}}{dz} - T_{zz} \frac{dv_x}{dz} \right) = \eta \frac{dv_x}{dz} \tag{1.2}$$

$$T_{xx} + \lambda_1 \left(v_0 \frac{dT_{xx}}{dz} - 2T_{xz} \frac{dv_x}{dz} \right) = 0, \qquad T_{zz} + \lambda_1 v_0 \frac{dT_{zz}}{dz} = 0 \tag{1.3}$$

We are required to find the solution of system (1.2), (1.3) in the domain $|z| \leqslant a$ (a plane channel) which satisfies the following boundary conditions.

We are given the longitudinal velocity $v_x(-a) = u_1$ and $v_x(a) = u_2$ (i.e. we are dealing with Couette-type flow). In addition, we are given the stresses T_{xx} and T_{zz} at the line of entry of the stream into the channel. For example, in the case of injection

($v_0 > 0$) we know that $T_{xx}(-a) = T_*$, $T_{zz}(-a) = T_0$, while with suction we have $T_{xx}(a) = T^*$, $T_{zz}(a) = T^\circ$.

The case $v_x(-a) = u_1$, $v_x(a) = u_2$ can be readily reduced to the case $v_x(-a) = 0$, $v_x(a) = u_2 - u_1 \equiv u$ by converting to a different inertial coordinate system.

The second equation of (1.3) has the solution

$$T_{zz} = T_0 \exp(-h - \zeta), \qquad \zeta(z) = z/\lambda_1 v_0, \qquad h = \zeta(a) \tag{1.4}$$

With a known distribution of T_{zz} values Eqs. (1.2) form a closed second-order system to which we must add two boundary conditions of "adhesion" of the longitudinal velocity. The first equation of (1.3) then enables us to find T_{xx}.

The second equation of (1.3) has the particular solution $T_{zz} \equiv 0$ corresponding to one of the two boundary conditions $T_0 = 0$ or $T^\circ = 0$. This solution is used by the authors of [2, 3] to consider the Couette flow described by model (1.1) with injection and suction. However, it is also possible to have a nonzero distribution $T_{zz}(z)$ which is consistent not only with the initial system of equations, but also with the following considerations.

For example, let the fluid be guided towards one of the walls by a system of capillaries in which there is fully developed flow along the z-axis. Considering the problem of Poiseuille flow in a circular capillary of radius r_0 with an impermeable boundary for model (1.1), we find that the distribution $T_{zz}(r)$ is not equal to zero, but is of the form

$$T_{zz} = \frac{1}{2}\frac{\lambda_1}{\eta} P^2 r^2, \qquad P = -\frac{\partial p}{\partial z} = \text{const} \qquad 0 \leqslant r \leqslant r_0$$

Moreover, the averages $\langle T_{zz}\rangle$ and $\langle v_z\rangle$ over the cross section of the capillary are related as follows:

$$\langle T_{zz}\rangle = 16\,\lambda_1\eta\langle v_z\rangle^2 / r_0^2$$

This means that specification of a positive T_{zz} at the line of entry of the fluid into the domain $|z| < a$ is physically permissible.

Substituting T_{zz} from (1.4) into (1.2), we obtain a first-order differential equation in T_{xz},

$$\lambda_1(c^2(z) - v_0^2)\frac{dT_{xz}}{dz} - v_0 T_{xz} = 0, \qquad c^2(z) = \frac{1}{\rho}\left(T_{zz} + \frac{\eta}{\lambda_1}\right) \tag{1.5}$$

The quantity $c^2(z)$ is the square of the speed of "transverse sound" and must be positive (otherwise the equations describing the small perturbations of the flow are nonevolutionary [4, 5]).

It is clear that the condition of evolutionary character of one-dimensional flows dependent on z is fulfilled for $T_{zz} > 0$.

The three-dimensional perturbations are evolutionary only if both $T_{zz} + \eta/\lambda_1 > 0$ and $(T_{xx} + \eta/\lambda_1)(T_{zz} + \eta/\lambda_1) > T_{xz}^2$ in the steady flow under consideration.

If the coefficient of the derivative in Eq. (1.5) does not vanish anywhere in the channel, i.e. if the transverse flow is either strictly subsonic or strictly supersonic, then the known value of the shear stress at one of the boundaries (e.g. at the lower boundary $T_{xz}(-a) = \tau_0$) can be used to find the distribution $T_{xz}(z)$. The quantity τ_0 in this case can be determined unambiguously from the given u and T_0 with the aid of the integral of the equation of motion projected onto the x-axis, which in the absence of a longitudinal pressure gradient and external body forces is given by

$$\rho v_0 v_x = T_{xz} - \tau_0$$

The resulting formulas for the distributions $T_{xz}(z)$ and $v_x(z)$ are

$$T_{xz} = \tau_0\left(\frac{\varkappa^{-1}e^{\zeta}+\sigma}{\varkappa^{-1}e^{-h}+\sigma}\right)^{\varkappa}, \qquad \sigma = \frac{T_{zz}(0)}{\rho v_0^2}, \qquad \varkappa = \left(\frac{c_0^2}{v_0^2}-1\right)^{-1}, \qquad c_0^2 = \frac{\eta}{\lambda_1\rho}$$

$$v_x = u\left\{\left(\frac{\varkappa^{-1}e^{\zeta}+\sigma}{\varkappa^{-1}e^{-h}+\sigma}\right)^{\varkappa}-1\right\}\left[\left(\frac{\varkappa^{-1}e^{h}+\sigma}{\varkappa^{-1}e^{-h}+\sigma}\right)^{\varkappa}-1\right]^{-1} \tag{1.6}$$

The relationship between τ_0 and the parameters of the problem is of the form

$$\rho v_0 u = \tau_0\left[\left(\frac{\varkappa^{-1}e^{h}+\sigma}{\varkappa^{-1}e^{-h}+\sigma}\right)^{\varkappa}-1\right] \tag{1.7}$$

Setting $\sigma = 0$ in (1.7) and taking the limit as $\lambda_1 \to 0$, we obtain the familiar formula for the analogous flows of a viscous fluid.

$$\rho v_0 u = \tau_0[\exp(2av_0/\nu) - 1], \qquad \nu = \eta/\rho$$

Now let us consider the case where the coefficient of dT_{xz}/dz in (1.5) vanishes at some point $z_0 \in (-a, a)$, i.e. where the transverse flow changes from subsonic to supersonic. The coordinate of the sonic line is given by the equation

$$\sigma\exp(-\zeta_0) = 1 - c_0^2/v_0^2, \qquad \zeta_0 = \zeta(z_0) = z_0/\lambda_1 v_0$$

The above equation has a solution in the interval $|\zeta| < |h|$ (under the condition $T_0 > 0$) if the parameters of the problem satisfy the following inequalities:

$$\rho(v_0^2 - c_0^2) < T_0 < \rho(v_0^2 - c_0^2)e^{2h}, \qquad v_0 > c_0$$
$$\rho(v_0^2 - c_0^2)e^{2h} < T_0 < \rho(v_0^2 - c_0^2), \qquad v_0 < -c_0$$

Equation (1.5) in this case can be written as

$$\varkappa^{-1}[1 - \exp(\zeta_0 - \zeta)]dT_{xz}/d\zeta - T_{xz} = 0$$

In the neighborhood of the point ζ_0 this equation can be replaced by

$$\varkappa^{-1}(\zeta - \zeta_0)dT_{xz}/d\zeta - T_{xz} = 0$$

The general solution of the latter equation is of the form

$$T_{xz} = C|\zeta - \zeta_0|^{\varkappa}$$

In the variable plane $T_{xz}\zeta$ the point $(0, \zeta_0)$ is a singular point of Eq. (1.5). Since $T_0 > 0$, it follows that $\varkappa < -1$, and the singular point is a saddle point. All of the integral curves $T_{xz}(\zeta)$ except the straight line $T_{xz} = 0$ go to infinity as ζ tends to ζ_0. A solution of the problem is then nonunique. The boundary conditions imposed on the longitudinal velocity $v_x(\zeta)$ can be satisfied for any given value of τ_0. In this case the quantity τ_0 cannot be found from a formula of the (1.7) type, and must be given. This means that the above problem has infinitely many solutions in the "transonic state" and that all of the solutions which satisfy the "adhesion" conditions for the longitudinal velocity at the channel boundaries have no physical meaning: the velocity v_x becomes infinite at the sonic line, and, since $\varkappa < -1$, the fluid leakage rate in the channel becomes infinite.

We note that if T_0 is allowed to have values in the interval $(-\eta/\lambda_1, 0)$, then the singular point is a node and the quantities T_{xz} and v_x have weak discontinuities at the sonic line. The solution in this case is nonunique, as in the case of a saddle point.

We have therefore established that a continuous transonic flow which satisfies the adhesion conditions for the longitudinal velocity at both channel boundaries cannot be constructed for $\varkappa < -1$. On the other hand, if we surrender the adhesion conditions at one of the walls, we can construct a unique continuous solution of the form $T_{xz}=0$, $v_x=$const.

Hence, the criterion of existence of a unique solution continuous throughout the channel is the absence of the second boundary condition. On the other hand, in the case of transonic flow with $\varkappa > 0$ two adhesion conditions for the longitudinal velocity turn out to be insufficient for determining the unique continuous solution, and it is necessary to specify a third boundary condition, e. g. $T_{xz}(-a) = \tau_0$.

A similar situation obtains in the case of one-dimensional unsteady flow. For example, let us assume that the steady flow under consideration is subject to the small perturbations δv_x, δT_{xz} which depend on z and t alone. Provided the speed of sound does not fluctuate, we have the following system of equations in δv_x, δT_{xz}:

$$\frac{\partial}{\partial t}\,\delta v_x + v_0\frac{\partial}{\partial z}\,\delta v_x - \frac{1}{\rho}\frac{\partial}{\partial z}\,\delta T_{xz} = 0 \tag{1.8}$$

$$\frac{\partial}{\partial t}\,\delta T_{xz} + v_0\frac{\partial}{\partial z}\,\delta T_{xz} - \left(T_{zz} + \frac{\eta}{\lambda_1}\right)\frac{\partial}{\partial z}\,\delta v_x + \frac{1}{\lambda_1}\,\delta T_{xz} = 0$$

The slope of the characteristics of system (1. 8) is determined by the equations

$$\frac{dz_+}{dt} = v_0 + c(z), \qquad \frac{dz_-}{dt} = v_0 - c(z), \qquad c(z) = \left[\frac{1}{\rho}\left(T_{zz} + \frac{\eta}{\lambda_1}\right)\right]^{1/2}$$

According to [5], system (1. 8) has a unique solution which is continuous in the rectangle R ($|z| \leqslant a$, $0 \leqslant t \leqslant T$) in the subsonic case if we specify one boundary condition for $z = -a$ and one condition for $z = a$.

In the case of supersonic flow we must either specify two boundary conditions at $z = -a$ if $v_0 > 0$ or two conditions at $z = a$ if $v_0 < 0$. The total number of boundary conditions in all these cases is two.

If the flow is transonic, then the straight line $z = z_0$ is the characteristic which separates the subsonic domain on the plane zt from the supersonic domain. In the case $\varkappa < -1$ it turns out that the subsonic domain lies to the right of z_0 for $v_0 < 0$ and to the left of z_0 for $v_0 > 0$. The total number of boundary conditions ensuring the uniqueness of the continuous solution is therefore one. In the case of transonic flow for $\varkappa > 0$ the number of boundary conditions is three, since the subsonic domain lies to the left of z_0 for $v_0 < 0$ and to the right of z_0 for $v_0 > 0$.

Analysis of model (1. 1) in the case of flow between coaxial cylinders moving at prescribed velocities along a common axis in the presence of a radial velocity v_r yields results similar to those obtained for a plane slit. Instead of Eq. (1. 5) we have

$$\lambda_1 (c^2(r) - v_r^2)\frac{d}{dr}(rT_{rz}) - v_r r T_{rz} = 0 \qquad v_r = \frac{Q}{2\pi r}$$

$$c^2(r) = \frac{1}{\rho}\left(T_{rr} + \frac{\eta}{\lambda_1}\right) = Cr^{-2}\exp\left(\frac{-\pi r^2}{\lambda_1 Q}\right) + \frac{\eta}{\lambda_1\rho}\left(1 - \frac{\lambda_1 Q}{\pi r^2}\right)$$

Continuous "transonic flow" in channels of the above type is also impossible to construct in the case of a longitudinal pressure gradient and $\varkappa < -1$. A similar situation obtains with other models of a Maxwellian fluid with finite elastic strains, e. g. with the equations of Oldroyd's "covariant" model [1] or of the De Witt model [6].

2. A solution of the above problem with a discontinuity of the second kind at the sonic line is physically unsatisfactory. It is therefore natural to attempt to construct a discontinuous but bounded solution of the problem. The introduction of finite discontinuities in a dissipative medium entails the imposition of additional dynamic conditions. Let us consider the equation of motion projected on the x-axis,

$$\rho v_0 \frac{dv_x}{dz} = \frac{dT_{xz}}{dz} + P + f_x \tag{2.1}$$

Here $P = -\partial p / \partial x$ and f_x is the density of the external body forces. Expressing the discontinuous distributions of the quantities v_x and T_{xz} as the limits of continuous distributions which change abruptly in the neighborhood of some point $z = \xi$, we find on the basis of the results of [7] that

$$\rho \vartheta_0 \{v_x\} - \{T_{xz}\} = F \tag{2.2}$$

The quantity F is defined as the limit

$$F = \lim_{\varepsilon \to +0} \int_{\xi-\varepsilon}^{\xi+\varepsilon} f_x dz$$

and represents the density of the external surface forces acting along the line $z = \xi$. It is also possible to define the surface moment of the external forces, which turns out to be

$$M = \lim_{\varepsilon \to +0} \int_{\xi-\varepsilon}^{\xi+\varepsilon} (z - \xi) f_x dz$$

Making use of the appropriate rheological equation (the second equation of (1.2)) for model (1.1) and of equation of motion (2.1), we find that

$$M = \lambda_1 \rho c^2 (\xi) \{v_x\} - \lambda_1 v_0 \{T_{xz}\} \tag{2.3}$$

The quantities F and M in two-constant model (1.1) represent the given external forces. It is not difficult to generalize formula (2.3) for the case of contact of two viscoelastic fluids.

Let us assume that the external surface forces are equal to zero at some point. We then have the following system of equations for determining the jumps $\{v_x\}$ and $\{T_{xz}\}$:

$$\rho v_0 \{v_x\} - \{T_{xz}\} = 0, \qquad \lambda_1 \rho c^2 \{v_x\} - \lambda_1 v_0 \{T_{xz}\} = 0$$

The determinant of this system is $\Delta = \lambda_1 \rho (c^2 - v_0^2)$. Hence, $\{v_x\} = 0$ and $\{T_{xz}\} = 0$ if at this point $c^2 \neq v_0^2$. If, on the other hand, $c^2 = v_0^2$, then (as we infer from Sect. 1) the jumps $\{v_x\}$ and $\{T_{xz}\}$ are equal either to zero or to infinity. We reject the latter alternative as physically meaningless.

Returning to the problem formulated in Sect. 1, we naturally assume that no external surface forces or moments are acting inside the channel, so that the solution must be continuous in the interval $|z| < a$. In the transonic case with $\varkappa < -1$ the distributions of T_{xz} and v_x inside the channel must be of the form $T_{xz} = 0$, $v_x = \text{const}$. This means that the boundary condition for the longitudinal velocity can only be satisfied by introducing the quantities F and M at the permeable walls. In the simplest case we can assume that the domain $|z| > a$ is occupied by the same viscoelastic fluid and approximate the channel boundaries by discontinuity surfaces at which the external forces and moments defined by formulas (2.2), (2.3) are given.

For example, let us consider "transonic injection", assuming that the entry line $z = -a$ is free from external forces and moments. This means that there is no velocity jump at the lower wall and that $v_x(z) = 0$ for $-a \leqslant z < a$. If the exit line $z = a$ is also free of external forces and moments, then $v_x(a) = 0$, and flow with prescribed boundary velocities is impossible. In order for $v_x(a) = u$ at the exit we need merely apply the force $F = \rho v_0 u$ and the moment $M = \lambda_1 \rho c^2 (a) u$ to the surface $z = a$.

We note that in the case of external surface forces and moments the solution has a discontinuity at the wall even if the injection velocity does not pass through the speed of sound inside the channel.

3. In real flows the discontinuity surfaces take the form of narrow zones characterized by abrupt changes in the flow parameters. One of the possible discontinuity structures arising at a wall confining a Maxwellian fluid stream is the boundary-layer flow of the three-constant model of a viscoelastic fluid with very small retardation times.

Let us consider the problem of Sect. 1 for an incompressible fluid whose behavior is described by the following rheological equations [1]:

$$p_{ij} = -p\delta_{ij} + T_{ij}, \qquad T_{ij} + \lambda_1 T_{ij}^{\cdot} = 2\eta\,(e_{ij} + \lambda_2 e_{ij}^{\cdot}) \tag{3.1}$$

$$e_{ij}^{\cdot} = \partial e_{ij}/\partial t + v_k e_{ij,k} - v_{i,k}e_{kj} - v_{j,k}e_{ik}$$

Here λ_1 is the relaxation time and λ_2 the retardation time. For this flow the first equation of (1.2) and the second equation of (1.3) remain unchanged for the three-constant model; the second equation of (1.2) is replaced by

$$T_{xz} + \lambda_1\left(v_0\frac{dT_{xz}}{dz} - T_{zz}\frac{dv_x}{dz}\right) = \eta\left(\frac{dv_x}{dz} + \lambda_2 v_0\frac{d^2v_x}{dz^2}\right) \tag{3.2}$$

This equation and the first equation of (1.2) and (1.4) imply that T_{xz} satisfy a differential equation which, unlike (1.5), is a second-order equation,

$$\frac{d^2T_{xz}}{d\zeta^2} + \mu\,\frac{c^2(\zeta) - v_0^2}{c_0^2}\,\frac{dT_{xz}}{d\zeta} - \mu\,\frac{v_0^2}{c_0^2}\,T_{xz} = 0, \qquad \zeta = \frac{z}{\lambda_1 v_0} \tag{3.3}$$

Equation (3.3) contains the parameter $\mu = \lambda_1/\lambda_2$. We must investigate the behavior of the solutions of this equation as μ tends to infinity.

Let us consider the case where Eq. (1.5) has a saddle point ζ_0 in the interval $|\zeta| < |h|$. It is clear that in the neighborhood of the point ζ_0 the behavior of the solutions of Eq. (3.3) is well described by the equation

$$\frac{d^2T_{xz}}{d\zeta^2} + \mu\left(1 - \frac{v_0^2}{c_0^2}\right)(\zeta - \zeta_0)\,\frac{dT_{xz}}{d\zeta} - \mu\,\frac{v_0^2}{c_0^2}\,T_{xz} = 0 \tag{3.4}$$

Equation (3.4) is an approximation of Eq. (3.3) for $|\zeta - \zeta_0| \ll 1$. Specifically, for $2a \ll \lambda_1 |v_0'|$ (the width of the channel is much smaller than the distance at which stress relaxation occurs) Eq. (3.4) can be applied throughout the channel. We make use of this fact in the discussion below.

Let us replace the unknown function by setting

$$T_{z} = \varphi(\zeta - \zeta_0)\exp\left[{}^1/_4\mu\,(v^2_0/c_0^2 - 1)\;(\zeta - \zeta_0)^2\right]$$

In this case $\varphi(\zeta)$ satisfies the equation

$$\frac{d^2\varphi}{d\zeta^2} - \left[\frac{1}{2}\mu\left(1 + \frac{v_0^2}{c_0^2}\right) + \frac{1}{4}\mu^2\left(\frac{v_0^2}{c_0^2} - 1\right)^2\zeta^2\right]\varphi = 0 \tag{3.5}$$

The general solution of this equation is of the form

$$\varphi = \zeta^{-1/2}y\,(\beta, {}^1/_4, \gamma\mu\zeta^2), \qquad \beta = \frac{1 + v_0^2/c_0^2}{4(1 - v_0^2/c_0^2)}, \qquad \gamma = \frac{1}{2}\left(\frac{v_0^2}{c_0^2} - 1\right)$$

$$y\,(k,\ m,\ x) = y\,(-k,\ m,\ -x) = C_1 M_{k,m}(x) + C_2 M_{k,-m}(x)$$

The Whittaker function $M_{k,m}(x)$ is given by the equation

$$M_{k,m}(x) = x^{m+1/2}\,e^{-1/2x}\,\Phi\,(m + {}^1/_2 - k,\ 2m + 1;\ x)$$

where $\Phi(a, b; x) \equiv {}_1F_1(a, b; x)$ is a degenerate Kummer hypergeometric function.

For Eq. (3.4) we have the boundary conditions

$$T_{xz}(-h) = \theta_0, \qquad T_{xz}(h) = \theta_0 + \rho v_0 u \tag{3.6}$$

which follow from the boundary conditions specified for the following third-order system closed with respect to v_x and T_{xz} and consisting of the first equation of (1.2) and Eq. (3.2):

$$v_x(-h) = 0, \qquad v_x(h) = u, \qquad T_{xz}(-h) = \theta_0$$

We emphasize that the specified shear stress $T_{xz}(-h) = \theta_0$ in the three-constant model is a natural boundary condition associated with the higher order of the model. The quantity θ_0 must either be determined by experiment or specified on the basis of additional physical assumptions. It is also possible for the quantity θ_0 to depend on the parameter μ.

A continuous solution of boundary value problem (3.6) for Eq. (3.4) exists for all values of u and θ_0. Let us write out this solution for the condition $\varkappa < -1$, assuming for simplicity that $\zeta_0 = 0$

$$T_{xz} = \frac{1}{2}\rho v_0 u \frac{\zeta}{h} \frac{\Phi(^3/_4 - \beta, {}^3/_2;\ \gamma\mu\zeta^2)}{\Phi(^3/_4 - \beta, {}^3/_2;\ \gamma\mu h^2)} + \left(\theta_0 + \frac{1}{2}\rho v_0 u\right) \frac{\Phi(^1/_4 - \beta, {}^1/_2;\ \gamma\mu\zeta^2)}{\Phi(^1/_4 - \beta, {}^1/_2;\ \gamma\mu h^2)} \tag{3.7}$$

To obtain the asymptotic form of the solution we use the formula

$$\Phi(a, b;\ x) = \frac{\Gamma(b)}{\Gamma(a)} x^{a-b} e^x (1 + O(x^{-1})), \qquad x \to +\infty$$

The asymptotic formula for T_{xz} as $\mu \to \infty$ then becomes

$$T_{xz} = \left|\frac{\zeta}{h}\right|^{\varkappa'} \exp\left(\gamma\mu(\zeta^2 - h^2)\right)\left(\theta_0 + \frac{1}{2}\rho v_0 u\left(1 + \operatorname{sign}\frac{\zeta}{h}\right)\right)(1 + O(\mu^{-1})) \tag{3.8}$$

$$|\zeta| \geqslant \varepsilon > 0, \qquad \varkappa' = (v_0^2 / c_0^2 - 1)^{-1} > 0$$

Let us introduce the notation $T_{xz}^*(\zeta) = \lim T_{xz}(\zeta, \mu)$ as $\mu \to \infty$. Expressions (3.7), (3.8) then imply that the function $T_{xz}^*(\zeta)$ is of the form

$$T_{xz}^*(-h) = \theta_0, \qquad T_{xz}^*(\zeta) = 0 \ (|\zeta| < |h|), \qquad T_{xz}^*(h) = \theta_0 + \rho v_0 u$$

For the function $v_x^*(\zeta) = \lim v_x(\zeta, \mu)$ as $\mu \to \infty$ we have

$$v_x^*(-h) = 0, \qquad v_x^*(\zeta) = -\theta_0/\rho v_0 \quad (|\zeta| < |h|), \qquad v_x^*(h) = u$$

As is evident from the above formulas, the limiting shear stress and longitudinal velocity distributions are discontinuous.

We note that the longitudinal velocity distribution of the three-constant model for a finite μ can also be achieved in the two-constant model. Let the functions $V_x(z)$, $T_{xz}(z)$ satisfy the first equation of (1.2) and Eq. (3.2). We set

$$v_x \equiv V_x, \qquad \tau_{xz} = T_{xz} - \frac{\eta}{\mu} \int_{-a}^{z} \exp\frac{y - z}{\lambda_1 v_0} V_x''(y)\, dy$$

The functions v_x, τ_{xz} satisfy the system of equations

$$\rho v_0 \frac{dv_x}{dz} = \frac{d\tau_{xz}}{dz} + f_x(z, \mu), \qquad \tau_{xz} + \lambda_1\left(v_0 \frac{d\tau_{xz}}{dz} - T_{zz}\frac{dv_x}{dz}\right) = \eta\frac{dv_x}{dz} \tag{3.9}$$

It is easy to see that the second equation of (3.9) coincides with the second equation of (1.2), and that the first equation of (3.9) is the equation of motion in the presence of body forces of a special type,

$$f_x(z, \mu) = \frac{\eta}{\mu}\left(V_x''(z) - \frac{1}{\lambda_1 v_0} \int_{-a}^{z} \exp\frac{y - z}{\lambda_1 v_0} V_x''(y)\, dy\right)$$

As μ tends to infinity the density of the body forces with a distribution of the form (3.8) increases without limit at $z = \pm a$. In the limiting case we can use the scheme described in Sect. 2, replacing the "retardation boundary layer" by a discontinuity surface. The discontinuous solution of the flow problem for the two-constant model is then uniquely determined by the quantities F and M. These quantities, e. g. at $z = -a$ can be computed from the formulas

$$F(-a) = \lim_{\varepsilon \to +0} \left(\lim_{\mu \to \infty} \int_{-a}^{-a+\varepsilon} f_x(z, \mu)\, dz \right), \qquad M(-a) = \lim_{\varepsilon \to +0} \left(\lim_{\mu \to \infty} \int_{-a}^{-a+\varepsilon} (z + a) f_x(z, \mu)\, dz \right)$$

In the narrow-channel approximation we obtain

$$F(-a) = \theta_0 \frac{a}{\lambda_1 v_0} \left(1 - \frac{c_0^2}{v_0^2}\right), \quad F(a) = (\theta_0 + \rho v_0 u) \frac{a}{\lambda_1 v_0} \left(1 - \frac{c_0^2}{v_0^2}\right), \quad M(\pm a) = 0$$

What we have said in this section implies that the infinite discontinuities described in Sect. 1 have no structure in the three-constant model. This fact indirectly confirms the physical meaninglessness of such discontinuities.

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BIBLIOGRAPHY

1. Oldroyd, J. G., On the formulation of rheological equations of state. Proc. Roy. Soc. (London), Vol. A200, №1063, 1950.
2. Datta, S. K., On the steady motion of an idealized elastico-viscous liquid through channels with suction and injection. J. Phys. Soc. Japan Vol. 16, №4, 1961.
3. Kaloni, P. N., Couette-type flow of a viscoelastic fluid through porous walls. J. Aeron. Sci. Vol. 29, №11, 1962.
4. Regirer, S. A. and Rutkevich, I. M., Certain singularities of the hydrodynamic equations of non-Newtonian media. PMM Vol. 32, №5, 1968.
5. Rutkevich, I. M., Some general properties of the equations of viscoelastic incompressible fluid dynamics. PMM Vol. 33, №1, 1969.
6. De Witt, T. W., Rheological equation of state which predicts non-Newtonian viscosity, normal stresses, and dynamic moduli. J. Appl. Phys. Vol. 26, №7, 1955.
7. Barenblatt, G. I. and Chernyi, G. G., On moment relations on surfaces of discontinuity in dissipative media. PMM Vol. 27, №5, 1963.

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